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Relativistic treatment of inertial spin effects

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In memory of the late Feza Gürsey

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Abstract. A relativistic spin operator for Dirac particles is identified and it is shown that a coupling of spin to angular velocity arises in the relativistic case, just as Mashhoon had speculated, and Hehl and Ni had demonstrated, in the non-relativistic case.

1. Introduction

It has been argued by Mashhoon [1] that there may exist a coupling of intrinsic spin with rotation, in analogy with the coupling of orbital angular momentum with rotation, which is the basis of the Sagnac effect. This new effect should, in principle, be observable in neutron interferometry and Mashhoon has outlined an experiment which could detect it.

The theoretical motivation offered by Mashhoon for this idea was rather general. Hehl and Ni, however, put it on a firmer footing by analysing the Dirac equation in a non-inertial frame of reference [2] (one subject to both acceleration and rotation) and in the non-relativistic limit of the Dirac Hamiltonian found the Mashhoon term, proportional to $\sigma \cdot \omega$, denoting coupling of intrinsic spin to angular velocity. The purpose of this paper is to point out that the Dirac Hamiltonian actually yields a term of the form $\boldsymbol{X} \cdot \omega$, where $\boldsymbol{X}$ is a *relativistic* spin operator; there is then a spin-angular velocity coupling at all energies and not just in the non-relativistic limit. In this limit, of course, $\boldsymbol{X}$ becomes $\sigma/2$, as expected.

The identification of a relativistic spin operator was a problem which received some attention in the 1960s and it turns out that $\boldsymbol{X}$ is in essence the Foldy–Wouthuysen 'mean spin operator' [3], as was first pointed out by Gürsey [4]. In fact, in their analysis referred to earlier, Hehl and Ni perform a Foldy–Wouthuysen (FW) transformation but treat this simply as a high grade way of taking a non-relativistic approximation. This is not to give the FW transformation its proper due. It is in fact valid at all energies; what it does is to separate the positive and negative energy sectors of the Dirac field. This is of course a sensible step to take prior to taking a non-relativistic approximation, but the validity of the FW transformation is not limited to this regime.

2. Mashhoon term–non-relativistic form

Hehl and Ni [2] write down the Dirac equation in a reference frame subjected both to acceleration $\boldsymbol{a}$ and to rotation ω. Expressing this in the form $i\hbar\partial\psi/\partial t = H\psi$ they find

$$H = \beta mc^2 + \mathrm{O} + \mathrm{E}$$

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$$\mathrm{O} = c\alpha \cdot \boldsymbol{p} + \frac{1}{2c}[(\boldsymbol{a}\cdot\boldsymbol{r})(\boldsymbol{p}\cdot\alpha) + (\boldsymbol{p}\cdot\alpha)(\boldsymbol{a}\cdot\boldsymbol{r})]$$
$$\mathrm{E} = \beta m(\boldsymbol{a}\cdot\boldsymbol{r}) - \omega\cdot(\boldsymbol{L}+\boldsymbol{S}). \tag{2.1}$$

It is the presence of the 'odd' term O, coupling the large and small components of the Dirac spinor, which necessitates an FW transformation. The Mashhoon term $\omega \cdot \boldsymbol{S}$, with $\boldsymbol{S} = (\hbar/2)\sigma$, appears in E. After three FW transformations the resulting Hamiltonian, in the non-relativistic approximation, is

$$H = \beta mc^2 + \frac{\beta}{2m}\boldsymbol{p}^2 + \beta m(\boldsymbol{a}\cdot\boldsymbol{r}) + \frac{\beta}{2mc^2}\boldsymbol{p}(\boldsymbol{a}\cdot\boldsymbol{r})\cdot\boldsymbol{p} - \omega\cdot(\boldsymbol{L}+\boldsymbol{S})$$
$$+\frac{\hbar}{4mc^2}\sigma\cdot(\boldsymbol{a}\times\boldsymbol{p}) + \text{ higher-order terms.} \tag{2.2}$$

It is easily seen from (2.1) and (2.2) that in the case of a frame subjected to rotation but not acceleration ($\boldsymbol{a} = \boldsymbol{0}$), the Mashhoon term $(\hbar/2)\sigma\cdot\omega$ is present. In the non-relativistic limit this is correctly interpreted as a coupling of spin with angular velocity, since the spin operator in this limit is $(\hbar/2)\sigma\times\mathbf{1}$ (a 4×4 matrix, $\mathbf{1}$ being the unit 2×2 matrix). However, the question of whether a spin-angular velocity coupling exists in the general case is left open.

It will be recalled that one of the original motivations of Foldy and Wouthuysen to look for a recasting of the Dirac equation was the fact that the free Dirac Hamiltonian (in an inertial frame)

$$H = \beta mc^2 + c\alpha\cdot\boldsymbol{p} \tag{2.3}$$

does not commute with σ, hence if the spin operator is proportional to σ, then spin is not a constant of motion. One of the key results of Foldy and Wouthuysen was to find what they called a 'mean spin operator'

$$\boldsymbol{X}(\boldsymbol{p}) = \frac{m}{2E}\sigma + \frac{\sigma\cdot\mathrm{p}}{2E(E+m)}\boldsymbol{p} + \frac{\mathrm{i}}{2E}\gamma^5\gamma^0\sigma\times\boldsymbol{p} \tag{2.4}$$

a complicated and nonlinear function of $\boldsymbol{p}$ which reduces to $\frac{1}{2}\sigma$ in the limit $\boldsymbol{p}\to\mathbf{0}$. $\boldsymbol{X}$ commutes with H and so corresponds to a conserved spin operator. The question we are faced with is the search for a relativistically covariant spin operator. This is the subject of the next section, where it is shown that the wanted operator is in fact the mean spin operator (2.4).

3. Covariant spin operator

A covariant operator is one that transforms covariantly under the homogeneous Lorentz group (HLG). The search for this operator cannot be separated from a consideration of basis states for representations of the group. There are two fundamental inequivalent representations of HLG

$$(\tfrac{1}{2},0)\colon \varphi_{\mathrm{R}} \to \exp\{(\mathrm{i}/2)\sigma\cdot(\theta - \mathrm{i}\lambda)\}\varphi_{\mathrm{R}}$$
$$(0,\tfrac{1}{2})\colon \varphi_{\mathrm{L}} \to \exp\{(\mathrm{i}/2)\sigma\cdot(\theta + \mathrm{i}\lambda)\}\varphi_{\mathrm{L}} \tag{3.1}$$

see, for example [5, 6]. The states φ_{L} and φ_{R} are so-called left- and right-handed 2-spinors, the three parameters θ correspond to a general rotation, the three parameters λ to a general 'pure' Lorentz transformation (Lorentz boost), and σ are of course the Pauli spin matrices.

Under a Lorentz boost ($\theta = \mathbf{0}$) from the rest frame $p = 0$, left- and right-spinors with momentum p may be written down on observing that $\lambda = \cosh^{-1}\gamma$.

$$\varphi'_{\mathrm{R}}(p) = \frac{E + m + \sigma \cdot p}{[2m(E+m)]^{1/2}} \varphi_{\mathrm{R}}(0) \tag{3.2}$$

$$\varphi'_{\mathrm{L}}(p) = \frac{E + m - \sigma \cdot p}{[2m(E+m)]^{1/2}} \varphi_{\mathrm{L}}(0). \tag{3.3}$$

Defining the 2×2 matrix

$$\mathsf{P} = E + \sigma \cdot \boldsymbol{p} \tag{3.4}$$

it is easily seen that equations (3.2) and (3.3) may be cast in the form

$$\varphi'_{\mathrm{R}}(p) = \left(\frac{\mathsf{P}}{m}\right)^{1/2} \varphi_{\mathrm{R}}(0) \tag{3.5}$$

$$\varphi'_{\mathrm{L}}(p) = \left(\frac{\mathsf{P}}{m}\right)^{-1/2} \varphi_{\mathrm{L}}(0). \tag{3.6}$$

A *general* Lorentz transformation Λ^{μ}_{ν} (which includes rotations as well as boosts) takes a momentum p^{μ} into $p'^{\mu} = \Lambda^{\mu}_{\nu} p^{\nu}$. Corresponding to Λ, which relates p and p', there is a 2×2 matrix L which relates P and P'. Since, from (3.4), $\det \mathsf{P} = m^2$, a Lorentz invariant, it is clear that we can put [7]

$$\mathsf{P}' = \mathsf{L}\mathsf{P}\mathsf{L}^{\dagger} \tag{3.7}$$

with

$$\det \mathsf{L} = 1. \tag{3.8}$$

We then have, under the action of this transformation

$$\varphi'_{\mathrm{R}}(p') = \mathsf{L}\varphi_{\mathrm{R}}(p) \tag{3.9}$$

which, combined with (3.5) and (3.7), gives

$$\begin{aligned} \varphi'_{\mathrm{R}}(p') &= \left(\frac{\mathsf{P}'}{m}\right)^{1/2} \varphi'_{\mathrm{R}}(0) = \mathsf{L}\varphi_{\mathrm{R}}(p) = \mathsf{L}\left(\frac{\mathsf{P}}{m}\right)^{1/2} \varphi_{\mathrm{R}}(0) \\ \varphi'_{\mathrm{R}}(0) &= \left(\frac{\mathsf{P}'}{m}\right)^{-1/2} \mathsf{L} \left(\frac{\mathsf{P}}{m}\right)^{1/2} \varphi_{\mathrm{R}}(0) \equiv \mathsf{U}\varphi_{\mathrm{R}}(0). \end{aligned} \tag{3.10}$$

The matrix U is unitary

$$\begin{aligned} \mathsf{U}^{\dagger}\mathsf{U} &= \left(\frac{\mathsf{P}}{m}\right)^{1/2} \mathsf{L}^{\dagger} \left(\frac{\mathsf{P}'}{m}\right)^{-1/2} \left(\frac{\mathsf{P}'}{m}\right)^{-1/2} \mathsf{L} \left(\frac{\mathsf{P}}{m}\right)^{1/2} \\ &= \mathsf{P}^{1/2}\mathsf{L}^{\dagger}(\mathsf{P}')^{-1}\mathsf{L}\mathsf{P}^{1/2} = \mathsf{P}^{1/2}\mathsf{P}^{-1}\mathsf{P}^{1/2} = 1 \end{aligned}$$

where (3.7) has been used; and therefore corresponds to *rotations*, i.e. belongs to $SU(2)$. Equation (3.10) is, in fact, Wigner's 'little group' representation of the Poincaré (inhomogeneous Lorentz) group, here portrayed as simply a representation of HLG. The realization that this step could be taken to generate a 'new' representation of HLG seems to date back to the work of Shaw [8].

The salient point about Wigner's enlargement of the homogeneous Lorentz group to the inhomogeneous (Poincaré) group [9] is that the two Casimir operators of the Poincaré group are, in essence, mass and spin, the quantities providing kinematic 'labels' for quantum systems. The 'little group' of the Poincaré group, defined to leave the 4-momentum of a state unchanged, provides the definition of spin. In the case of states with timelike momenta the little group is $SU(2)$, as seen above.

The transformation law (3.10) certainly provides a representation of the Lorentz group, but it is a complicated one; the matrix $\mathbf{U}$ depends on the momentum p. The representation (3.10) is not covariant, whereas the representation (3.5) is. With this in mind we now proceed to find the covariant spin operator. In the zero-momentum limit it is $\sigma/2$: this is in the basis $\varphi_{\mathrm{R}}(0)$. In the basis (see (3.5))

$$\varphi_{\mathrm{R}}(p) = \left(\frac{\mathbf{P}}{m}\right)^{1/2} \varphi_{\mathrm{R}}(0) \equiv V\varphi_{\mathrm{R}}(0) \tag{3.11}$$

it is

$$V^{-1}\frac{\sigma}{2}V = \frac{1}{2m(E+m)}(E+m-\sigma\cdot\boldsymbol{p})\frac{\sigma}{2}(E+m+\sigma\cdot\boldsymbol{p}). \tag{3.12}$$

We now enlarge to a four-dimensional basis

$$\begin{aligned}\varphi(p) = \begin{pmatrix}\varphi_{\mathrm{R}}(p)\\ \varphi_{\mathrm{L}}(p)\end{pmatrix} &= \frac{1}{[2m(E+m)]^{1/2}}\begin{pmatrix}E+m+\sigma\cdot\boldsymbol{p} & 0\\ 0 & E+m-\sigma\cdot\boldsymbol{p}\end{pmatrix}\begin{pmatrix}\varphi_{\mathrm{R}}(0)\\ \varphi_{\mathrm{L}}(0)\end{pmatrix}\\ &= \frac{1}{[2m(E+m)]^{1/2}}(E+m+\gamma_5\sigma\cdot\boldsymbol{p})\varphi(0).\end{aligned} \tag{3.13}$$

Using the relations $\gamma_5\sigma^i = \alpha^i = \gamma^0\gamma^i = -\gamma^i\gamma^0$ it follows that

$$\gamma_5\sigma^i\left(\frac{1+\gamma^0}{2}\right) = -\gamma^i\left(\frac{1+\gamma^0}{2}\right)$$

and thus restricted to *positive energy states*, we have

$$\begin{aligned}\varphi^{(+)}(p) &= \frac{1}{[2m(E+m)]^{1/2}}(E+m+\gamma_5\sigma\cdot\boldsymbol{p})\varphi^{(+)}(0)\\ &= \frac{1}{[2m(E+m)]^{1/2}}(E+m-\gamma\cdot\boldsymbol{p})\varphi^{(+)}(0) = \left[\frac{E}{m}\right]^{1/2} U(p)\varphi^{(+)}(0)\end{aligned} \tag{3.14}$$

where

$$U(p) = \frac{1}{[2E(E+m)]^{1/2}}(E+m-\gamma\cdot\boldsymbol{p}) \tag{3.15}$$

is a unitary operator. It is in fact the Foldy–Wouthuysen operator [3]. In the basis $\varphi^{(+)}(0)$ the spin operator is $\sigma/2$, so in the basis $\varphi^{(+)}(p)$ it is

$$\begin{aligned}\boldsymbol{X} = U^{\dagger}(p)\frac{\sigma}{2}U(p) &= \frac{1}{2E(E+m)}(E+m+\gamma\cdot\boldsymbol{p})\frac{\sigma}{2}(E+m-\gamma\cdot\boldsymbol{p})\\ &= \frac{m}{E}\frac{\sigma}{2} + \frac{\sigma\cdot\boldsymbol{p}}{2E(E+m)}\boldsymbol{p} + \frac{\mathrm{i}}{2E}\gamma^5\gamma^0\sigma\times\boldsymbol{p}\end{aligned} \tag{3.16}$$

which is the Foldy–Wouthuysen *mean spin operator* (2.4).

We conclude that the relativistic spin operator is the same as the FW mean spin operator, when applied to the spectrum of *positive energy states*. This observation was first made by Gürsey [4].

4. Relativistic Mashhoon term

Reverting to the original problem, the Dirac Hamiltonian in a frame rotating with angular velocity ω is, from (2.1),

$$H = \gamma^0 mc^2 + c\alpha \cdot \boldsymbol{p} - \omega \cdot (\boldsymbol{L} + \boldsymbol{S}) \tag{4.1}$$

with $\boldsymbol{S} = \sigma/2$. This Hamiltonian suffers from the problems highlighted by Foldy and Wouthuysen; that the eigenvalues of the velocity operator are (in the absence of notation) $\pm c$ and that spin is not conserved. These problems are resolved on converting the Hamiltonian to block diagonal form and projecting out the positive energy states. The *exact* form of the Foldy–Wouthuysen (as distinct from the approximate one used by Hehl and Ni) is given by equation (3.15). This transformation has the effect not only of changing the spin operator $\boldsymbol{S}$ into $\boldsymbol{X}$, given by (3.16), but also of changing the position operator $\boldsymbol{r}$ into the 'mean position operator' $\boldsymbol{R}$ [3] and hence the orbital angular monentum $\boldsymbol{L}$ into the 'mean orbital angular momentum' $\boldsymbol{M} = \boldsymbol{R} \times \boldsymbol{p}$, so that the resulting Hamiltonian is

$$H' = \gamma^0 E - \omega \cdot (\boldsymbol{M} + \boldsymbol{X}). \tag{4.2}$$

The last term in (4.2) is the relativistic Mashhoon term.

5. Conclusion

The coupling of spin to angular velocity in a rotating frame has been shown, in the case of spin-$\frac{1}{2}$ particles, to have a formulation consistent with relativity. The relevant relativistic spin operator is essentially the Foldy–Wouthuysen mean spin operator.

Dedication

This paper is dedicated to the memory of Feza Gürsey, under whose guidance the author was made familiar with the problem of relativistic spin operators in the 1960s.

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